Concentration Inequalities for Random Matrices

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exponential tail inequalities

classical theme in probability and statistics

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quantify the asymptotic statements

exponential tail inequalities

classical theme in probability and statistics

quantify the asymptotic statements

central limit theorems

large deviation principles

classical exponential inequalities

sum of independent random variables

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

classical exponential inequalities

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$0 \leq X_i \leq 1$ independent

$$\mathbb{P}\big(S_n \geq \mathbb{E}(S_n) + t\big) \leq e^{-t^2/2}, \quad t \geq 0$$

Hoeffding's inequality

classical exponential inequalities

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Hoeffding's inequality

same as for X_i standard Gaussian

central limit theorem

measure concentration ideas

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asymptotic geometric analysis

V. Milman (1970)

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$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

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Gaussian sample

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M. Talagrand (1995)

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$$\mathbb{P}\big(|Z - M| \geq t\big) \leq C \exp\left(-\frac{t}{C} \log\left(1 + \frac{t}{\sigma^2 + M}\right)\right), \quad t \geq 0$$

$C > 0$ numerical constant, M mean or median of Z

$\sigma^2 = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{E}(f^2(X_i))$

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P.-M. Samson (2000) (dependence)

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numerous applications

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- geometric functional analysis
- discrete and combinatorial probability
- empirical processes
- statistical mechanics
- random matrix theory

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recent studies of

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random matrices, longest increasing subsequence,

random growth models, last passage percolation...

sample covariance matrices

multivariate statistical inference

principal component analysis

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population $(Y_1, \ldots, Y_N)$

Y_j vectors (column) in $\mathbb{R}^M$ (characters)

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(independent) **Gaussian** Y_j : **Wishart matrix models**

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population covariance matrix

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$$\frac{M}{N} \sim \rho \in (0, \infty) \qquad N \to \infty$$

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$$Y = (Y_{ij})_{1 \le i \le M, 1 \le j \le N}$$

Y_{ij} independent identically distributed

(real or complex)

$$\mathbb{E}(Y_{ij}) = 0, \quad \mathbb{E}(Y_{ij}^2) = 1$$

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numerous extensions

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of $\quad Y\,Y^t \quad$ ($M \times M$ non-negative symmetric matrix)

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asymptotics $\quad M = M(N) \sim \rho\, N \qquad N \to \infty$

Marchenko-Pastur theorem (1967)

asymptotic behavior of the spectral measure $(\widehat{\lambda}_k^N = \lambda_k^N / N)$

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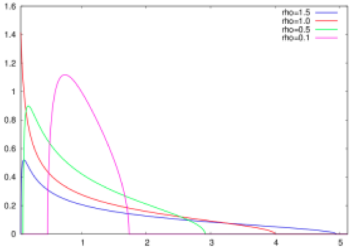

Marchenko-Pastur theorem

$$\frac{1}{M} \sum_{k=1}^{M} \delta_{\widehat{\lambda}_k^N} \ \to \ \nu \quad \text{on} \quad \big(a(\rho), b(\rho)\big) \qquad M \sim \rho N$$

global regime

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fluctuations of the spectral measure

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fluctuations of the spectral measure

$$\sum_{k=1}^{M}\Big[f\big(\widehat{\lambda}_k^N\big) - \int_{\mathbb{R}} f\, d\nu\Big] \to G \quad \text{Gaussian variable}$$

$f : \mathbb{R} \to \mathbb{R}$ smooth

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behavior of the individual eigenvalues

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local regime

behavior of the individual eigenvalues

spacings (bulk behavior)

extremal eigenvalues (edge behavior)

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N$

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$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N}$$

extremal eigenvalues

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$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \;\to\; b(\rho) = \big(1 + \sqrt{\rho}\,\big)^2 \qquad M \sim \rho\, N$$

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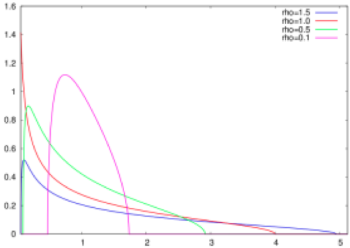

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fluctuations around $b(\rho)$

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fluctuations around $b(\rho)$

complex or real Gaussian (Wishart matrices)

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fluctuations around $b(\rho)$

complex or real Gaussian (Wishart matrices)

$$M^{2/3} \big[\, \widehat{\lambda}_M^N - b(\rho) \big] \ \to \ C(\rho)\, F_{\rm TW}$$

extremal eigenvalues

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$$M^{2/3} N^{-1} \big[\lambda_M^N - b(\rho) N\big] \;\to\; C(\rho)\, F_{\mathrm{TW}}$$

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N$

$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \ \rightarrow \ b(\rho) = \left(1 + \sqrt{\rho}\,\right)^2 \qquad M \sim \rho\, N$$

fluctuations around $b(\rho)$

complex or real Gaussian (Wishart matrices)

$$M^{2/3} N^{-1} \left[\lambda_M^N - b(\rho) N\right] \rightarrow C(\rho)\, F_{\mathrm{TW}}$$

F_{TW} **C. Tracy, H. Widom (1994)** distribution

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(complex) $$F_{\mathrm{TW}}(s) = \exp\left(- \int_s^\infty (x-s) u(x)^2 dx \right), \quad s \in \mathbb{R}$$

$u'' = 2u^3 + xu$ Painlevé II equation

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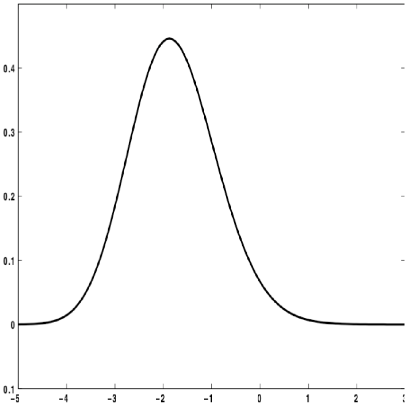
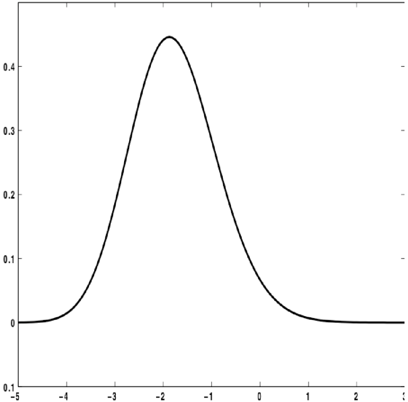


density

mean $\simeq -1.77$

$$F_{\mathrm{TW}}(s) \sim e^{-s^3/12} \quad \text{as} \quad s \to -\infty$$

$$1 - F_{\mathrm{TW}}(s) \sim e^{-4s^{3/2}/3} \quad \text{as} \quad s \to +\infty$$



density (similar for real case)

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Gaussian (Wishart matrices)

Gaussian (Wishart matrices)

completely solvable models

Gaussian (Wishart matrices)

completely solvable models

determinantal structure

orthogonal polynomial analysis

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asymptotics of Laguerre orthogonal polynomials

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extension to non-Gaussian matrices

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A. Soshnikov (2001-02)

moment method $\mathbb{E}\big(\mathrm{Tr}\big((YY^t)^p\big)\big)$

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local Marchenko-Pastur law

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Lindeberg comparison method

symmetric matrices

(brief) survey of recent approaches to

non-asymptotic exponential inequalities

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quantify the limit theorems

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quantify the limit theorems

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catch the **new rate** $(\mathrm{mean})^{1/3}$

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from the Gaussian case to non-Gaussian models

two main questions and objectives

two main questions and objectives

tail inequalities for the spectral measure

$$\mathbb{P}\Big(\sum_{k=1}^{M} f(\widehat{\lambda}_k^N) \geq t\Big)$$

Marchenko-Pastur theorem

$$\frac{1}{M}\sum_{k=1}^{M}\delta_{\widehat{\lambda}_k^N} \ \to \ \nu \quad \text{on} \quad \big(a(\rho), b(\rho)\big) \qquad M \sim \rho\, N$$

global regime

large deviation asymptotics of the spectral measure

fluctuations of the spectral measure

$$\sum_{k=1}^{M}\Big[f\big(\widehat{\lambda}_k^N\big) - \int_{\mathbb{R}} f\, d\nu\Big] \ \to \ G \quad \text{Gaussian variable}$$

$f : \mathbb{R} \to \mathbb{R}$ smooth

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satisfactory for the global regime

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satisfactory for the global regime

less satisfactory for the local regime

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$$F = F(Y\,Y^t) = F(Y_{ij})$$

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specific functionals

eigenvalue counting function

extreme eigenvalues

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with respect to the Euclidean structure on $M \times M$ matrices

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convex if f is convex

concentration inequalities

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ $\quad 1$-Lipschitz

$X_1, \ldots, X_n$ independenty standard Gaussian

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}\big(F(X)\big) + t \Big) \leq e^{-t^2/2}, \quad t \geq 0$$

$0 \leq X_i \leq 1$ independent, F 1-Lipschitz and convex

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}\big(F(X)\big) + t \Big) \leq 2\, e^{-t^2/4}, \quad t \geq 0$$

M. Talagrand (1995)

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Gaussian entries Y_{ij}

$f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2)$ 1-Lipschitz

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compactly supported entries Y_{ij}

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non-Lipschitz functions f

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typically $f = \mathbf{1}_I$, $I \subset \mathbb{R}$ interval

$$\sum_{k=1}^{M} f\big(\widehat{\lambda}_k^N\big) = \#\big\{\widehat{\lambda}_k^N \in I\big\} = \mathcal{N}_I$$ counting function

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Wishart matrices (determinantal structure)

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I interval in (a, b)

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comparison with Wishart model

non-Gaussian covariance matrices

comparison with Wishart model

partial results

localization results **L. Erdös, H.-T. Yau (2009-12)**

Lindeberg comparison method **T. Tao, V. Vu (2010-11)**

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S. Dallaporta, V. Vu (2011)

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T. Tao, V. Vu (2012)

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finite M inequalities

at the $(\mathrm{mean})^{1/3}$ rate

reflecting the tails of F_{TW}

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bounds on $\mathrm{Var}(\,\widehat{\lambda}_M^N)$

measure concentration tool

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(Gaussian) Wishart matrix $Y\,Y^t$

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Gaussian concentration

$$\mathbb{P}\Big(\,\widehat{s}_M^N \ge \mathbb{E}\big(\widehat{s}_M^N\big) + t\Big) \le e^{-M\,t^2/C}, \quad t \ge 0$$

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correct large deviation bounds $(t \geq 1)$

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does not fit the small deviation regime $t = s\,M^{-2/3}$

extreme eigenvalues

alternate tools

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Riemann-Hilbert analysis (Wishart matrices)

tri-diagonal representations (Wishart and β-ensembles)

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bounds for Wishart matrices

tri-diagonal representation

B. Rider, M. L. (2010)

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fit the **Tracy-Widom** asymptotics $\quad (\varepsilon = s\, M^{-2/3})$

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fit the **Tracy-Widom** asymptotics $\quad (\varepsilon = s\,M^{-2/3})$

$$1 - F_{\mathrm{TW}}(s) \sim e^{-s^{3/2}/C} \quad (s \to +\infty)$$

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$$b(\rho) = \left(1 + \sqrt{\rho}\right)^2$$

$$\widehat{\lambda}_M^N = \lambda_M^N / N, \qquad M = M(N) \sim \rho\, N$$

$$\frac{(\sqrt{MN})^{1/3}}{(\sqrt{M} + \sqrt{N})^{4/3}}\left(\lambda_M^N - (\sqrt{M} + \sqrt{N})^2\right) \;\to\; F_{\mathrm{TW}}$$

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$$N+1 \geq M \qquad 0 < \varepsilon \leq 1$$

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$$\mathbb{P}\Big(\lambda_M^N \leq (\sqrt{M}+\sqrt{N})^2(1-\varepsilon)\Big) \;\leq\; C\,e^{-MN\,\varepsilon^{3}\left(\frac{1}{\varepsilon}\wedge\left(\frac{M}{N}\right)^{1/2}\right)/C}$$

bi and tri-diagonal representation

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$$B = \begin{pmatrix} \chi_N & 0 & 0 & \cdots & \cdots & 0 \\ \widetilde{\chi}_{(M-1)} & \chi_{N-1} & 0 & 0 & \cdots & \vdots \\ 0 & \widetilde{\chi}_{(M-2)} & \chi_{N-3} & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \widetilde{\chi}_2 & \chi_{N-M+2} & 0 \\ 0 & \cdots & \cdots & 0 & \widetilde{\chi}_1 & \chi_{N-M+1} \end{pmatrix}$$

$\chi_{(N-1)}, \ldots, \chi_1, \quad \widetilde{\chi}_{(M-1)}, \ldots, \widetilde{\chi}_1$ independent chi-variables

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\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \cdots & \ddots & \widetilde{\chi}_2 & \chi_{N-M+2} & 0 \\
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extension to β-ensembles

bounds for non-Gaussian entries

moment method $\mathbb{E}\big(\mathrm{Tr}\big((YY^t)^p\big)\big)$

O. Feldheim, S. Sodin (2010)

bounds for **non-Gaussian entries**

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largest eigenvalue (symmetric, subGaussian entries)

$$\mathbb{P}\big(\,\widehat{\lambda}_M^N \geq b(\rho) + \varepsilon\big) \leq C\, e^{-M\,\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

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necessary for variance bounds

variance level

$$\mathrm{Var}(\widehat{\lambda}_M^N) = O\Big(\frac{1}{M^{4/3}}\Big)$$

S. Dallaporta (2012)

variance level

$$\mathrm{Var}(\widehat{\lambda}_M^N) = O\Big(\frac{1}{M^{4/3}}\Big)$$

S. Dallaporta (2012)

comparison with Wishart model

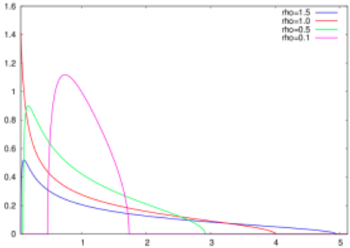
localization results **L. Erdös, H.-T. Yau (2009-12)**

Lindeberg comparison method **T. Tao, V. Vu (2010-11)**

smallest eigenvalue

soft edge $M = M(N) \sim \rho\, N, \quad \rho < 1$

$$a(\rho) = \left(1 - \sqrt{\rho}\,\right)^2$$



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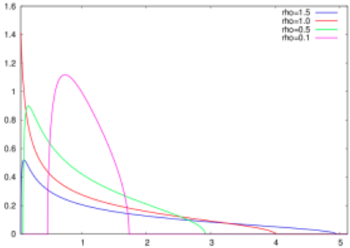

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Wishart matrices **B. Rider, M. L. (2010)**

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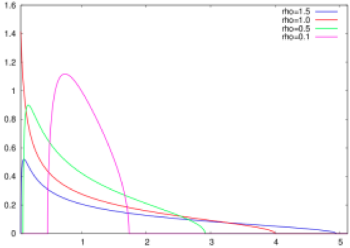
hard edge $M = N, \quad \rho = 1$

$$a(\rho) = \left(1 - \sqrt{\rho}\right)^2 = 0$$



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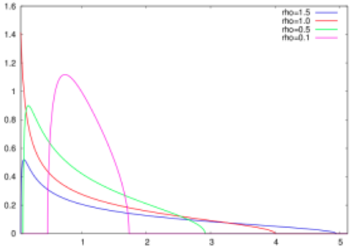

$$\mathbb{P}\Big(\widehat{\lambda}_1^N \leq \frac{\varepsilon}{N^2}\Big) \leq C\sqrt{\varepsilon} + C\,e^{-cN}$$

large families of covariance matrices

M. Rudelson, R. Vershynin (2008-10)