Statistical Estimation of Optimal Portfolios for Dependent Returns

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Introduction

- Optimal portfolios are determined by the mean μ and variance Σ of the portfolio return.

- Several authors proposed estimators of the optimal portfolios as the functions of the sample mean $\hat{\mu}$ and the sample variance $\hat{\Sigma}$ for independent returns of assets.

- However, many empirical studies show that return processes are generally **non-Gaussian** and **dependent**.

Introduction

- Basak, Jagannathan and Sun (2002) showed the consistency of optimal portfolio estimators when the portfolio returns are stationary processes.

- In the literature there has been no study on the **asymptotic efficiency** of estimators for optimal portfolios.

- In this presentation, denoting the optimal portfolios by a function $g = g(\mu , \Sigma)$ of μ and $\sum$, we discuss the **asymptotic efficiency** of estimators $\hat{g} = g(\hat{\mu} , \hat{\Sigma})$ when the returns are **Vector-valued Non –Gaussian (Locally) Stationary Process**.

Optimal Portfolios

- Asset Returns and Portfolio Weights
  - $i, (i = 1, \ldots, m)$: finite number of assets
  - $X(t) = \left(X_1(t), \ldots, X_m(t)\right)'$: the random returns on m assets at time t with $\mu = E\{X(t)\}$ and $\Sigma = Cov\left(X(t)\right)$
  - $\alpha = (\alpha_1, \ldots, \alpha_m)$: the vector of portfolio weights
- Portfolio Quantities
  - $X(t)'\alpha$: the return of portfolio
  - $\mu(\alpha) = \mu'\alpha$: the expectation of $X(t)'\alpha$
  - $\eta^2(\alpha) = \alpha'\Sigma\alpha$: the variance of $X(t)'\alpha$

Optimal Portfolios

- **Mean-Variance Optimal Portfolio**

Optimization：

$$\begin{cases} \max_{\alpha} \left\{ \mu(\alpha) - \beta\eta^2(\alpha) \right\} \\ subject\ \ to\ \ e'\alpha = 1 \end{cases}$$

where $e = (1,\ldots,1)'\ (m \times 1 - vector)$, and β is a given positive number.

The solution is

$$g(\mu,\Sigma) = \frac{1}{2\beta}\left\{ \Sigma^{-1}\mu - \frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e}\Sigma^{-1}e \right\} + \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$$

Asymptotic Theory for Fundamental Quantities

Return Process

$\{X(t)=(X_1(t),\ldots,X_m(t))' \,;\, t\in\mathbf{Z}\}$ is an *m*-vector linear process

$$X(t)=\sum_{j=0}^{\infty}A(j)U(t-j)+\mu, \qquad t\in\mathbf{Z}$$

where $U(t)$'s are

- i.i.d. m-vector random variables
- $E\{U(t)\}=0, \; Cov\{U(t)\}=K$
- $fourth\ order\ cumulants<\infty$

Asymptotic Theory for Fundamental Quantities

- **Parameter**

$$\theta = \left(\mu', vech\left\{R(0)\right\}\right)'$$

where $R(0) = E\{X(t)-\mu\}\{X(t)-\mu\}'$,

- **Estimator**

Given $\{X(1), \ldots, X(n)\}$, we construct

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} X(t),$$

$$\hat{R}(0) = \frac{1}{n}\sum_{t=1}^{n}\{X(t)-\hat{\mu}\}\{X(t)-\hat{\mu}\}',$$

$$\hat{\theta} = \left(\hat{\mu}', vech\left\{\hat{R}(0)\right\}\right)'$$

Asymptotic Theory for Fundamental Quantities

CLT for Non-Gaussian Stationary Process

- Theorem 1

$$\sqrt{n}\left(\hat{\theta} - \theta \right) \xrightarrow{\Lambda} N\left(0,\ \Omega_{NG}\right)$$

- Theorem2

$$\sqrt{n}\left(g\left(\hat{\theta}\right) - g\left(\theta\right) \right) \xrightarrow{\Lambda} N\left(0,\ \left(\frac{\partial g}{\partial \theta'}\right) \Omega_{NG} \left(\frac{\partial g}{\partial \theta'}\right)' \right)$$

Asymptotic Efficiency of Estimators of Optimal Portfolios

- CLT for **Gaussian** Stationary Process
  - Theorem 2'

$$\sqrt{n}\left(g\left(\hat{\theta}\right) - g(\theta) \right) \xrightarrow{\Lambda} N\left(0, \left(\frac{\partial g}{\partial \theta'}\right)\Omega_G\left(\frac{\partial g}{\partial \theta'}\right)' \right)$$

  - We evaluate

$$\boxed{\mu'\left(V_{NG} - V_G \right)\mu}$$

where

$$V_{NG} = \left(\frac{\partial g}{\partial \theta'}\right)\Omega_{NG}\left(\frac{\partial g}{\partial \theta'}\right), \quad V_G = \left(\frac{\partial g}{\partial \theta'}\right)\Omega_G\left(\frac{\partial g}{\partial \theta'}\right)$$

$$g(\mu,\Sigma)=\frac{1}{2\beta}\Sigma^{-1}(\mu - R_0 e)$$

Asymptotic Efficiency of Estimators of Optimal Portfolios

- VMA(1) model：

$$X(t) = \begin{pmatrix} 1-0.4B & 0 \\ 0 & 1-0.6B \end{pmatrix} U(t) + \begin{pmatrix} 0.1 \\ 0.3 \end{pmatrix}$$

$$U(t) = \begin{pmatrix} u_1(t) - \kappa_1 \\ u_2(t) - \kappa_2 \end{pmatrix}$$

$$u_i(t) \sim i.i.d.\ EX(\kappa_i)\ \ (i = 1,2)$$

Asymptotic Efficiency of Estimators of Optimal Portfolios

- Table ：VMA(1) model

$\kappa_1 \mid \kappa_2$	1.0	2.0	3.0	4.0	5.0
1.0	-0.00978	-0.00433	-0.00270	-0.00221	-0.00202
2.0	-0.00823	-0.00278	-0.00115	-0.00066	-0.00047
3.0	-0.00806	-0.00261	-0.00098	-0.00049	-0.00030
4.0	-0.00802	-0.00257	-0.00093	-0.00044	-0.00025
5.0	-0.00800	-0.00255	-0.00092	-0.00043	-0.00024

Asymptotic Efficiency of Estimators of Optimal Portfolios

- The necessary and sufficient condition for Efficient Portfolio when the return process is **Gaussian Stationary Process**.
  - Theorem 3

$$g\left(\hat{\theta}\right) \text{ is asymptotically efficient}$$

$$\Leftrightarrow$$

$$\exists \text{ a matrix "}C\text{" (independent of } \lambda) \text{ s.t.}$$

$$\left\{f_\eta(\lambda)' \otimes f_\eta(\lambda)\right\}\Phi$$
$$= \left(vec\left\{\partial f_\eta(\lambda)/\partial\eta_1\right\},\ldots,vec\left\{\partial f_\eta(\lambda)/\partial\eta_q\right\}\right)C$$

where $f_\eta(\lambda)$ is a spectral density matrix of the return process.

Asymptotic Efficiency of Estimators of Optimal Portfolios

- V-MA model
- V-ARMA(p_1,p_2) model (with $p_1<p_2$)

$$X(t) - \Theta_1 X(t-1) - \dots - \Theta_{p_1} X(t-p_1)$$

$$= \varepsilon(t) - \Psi_1 \varepsilon(t-1) - \dots - \Psi_{p_2} \varepsilon(t-p_2)$$

- V-Exponential model

$\Rightarrow$ **Not Asymptotically Efficient**

Asymptotic Efficiency of Estimators of Optimal Portfolios

- We evaluate

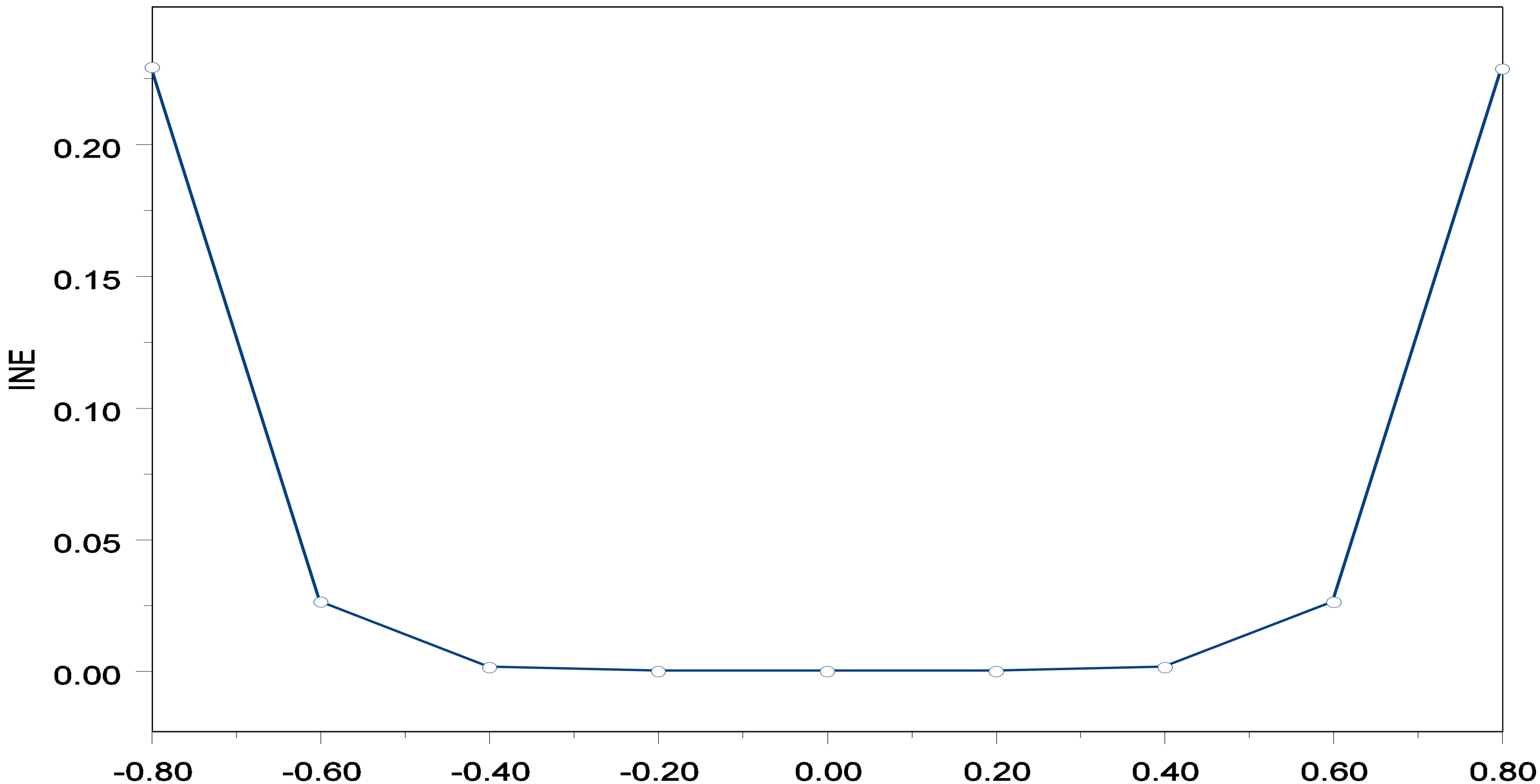
$$INE \equiv \det\left[\left\{\textbf{Asymptotic variance } of \ \hat{\theta}\right\} - \boldsymbol{\Phi}(\eta)^{-1}\right]$$

for VMA(1) model

$$X(t) = \begin{pmatrix} 1-\eta_1 B & 0 \\ 0 & 1-\eta_1 B \end{pmatrix} U(t)$$

$$U(t) \overset{i.i.d.}{\sim} N\left(0, \begin{pmatrix} \eta_2 & \eta_3 \\ \eta_3 & \eta_2 \end{pmatrix}\right)$$

Asymptotic Efficiency of Estimators of Optimal Portfolios



$$\eta_1 = -0.8(0.2)0.8,\ \eta_2 = 0.5, \eta_3 = 0.1$$

Construction of Efficient Estimators

Conditional Heteroscedastic Autoregressive Nonlinear Model (CHARN model)

Suppose that $\{X(t)\}$ is generated by

$$X(t) = F_\theta(X(t-1), \ldots, X(t-p_1), \mu) + H_\theta(X(t-1), \ldots, X(t-p_2))\, U(t),$$

where

- F_θ vector-valued measurable function
- H_θ matrix-valued measurable function
- $U(t)$ i.i.d. random vectors with p.d.f $p(\cdot)$

Construction of Efficient Estimators

Efficient Estimator for CHARN model

$$\hat{\theta}_{ML} \equiv \arg\max_{\theta} \Lambda_n(\theta_0, \theta)$$

where

$$\Lambda_n(\theta_0, \theta) = \sum_{t=p}^{n} \log \frac{p\left\{H_\theta^{-1}(X(t) - F_\theta)\right\} \det H_\theta}{p\left\{H_{\theta_0}^{-1}(X(t) - F_{\theta_0})\right\} \det H_{\theta_0}}$$

and $\theta_0 \in \Theta$ is some fixed value.

Construction of Efficient Estimators

Efficient Portfolio Estimator

$\hat{\theta}_{ML}$ *is asymptotically efficient* (by Kato et al (2006))

$$\Rightarrow \quad \text{Write} \quad \mu_\theta = E(X(t)) = F_\theta \ ,$$

$$\Sigma_\theta = Cov(X(t)) = H_\theta H_\theta '$$

$$\text{and} \quad g(\theta) = g((\mu_\theta ', vech\Sigma_\theta ')').$$

Then,

$$g\left(\hat{\theta}_{ML}\right) \ \text{is asymptotically efficient}.$$

Locally Stationary Returns

Locally Stationary Return

Suppose that $\{X(t,n); t=1,\ldots,n, n\in \boldsymbol{N}\}$ is generated by

$$X(t,n)=\mu\left(\frac{t}{n}\right)+\int_{-\pi}^{\pi}\exp\ (i\lambda t)A^{\circ}(t,n,\lambda)d\xi(\lambda).$$

Then, we construct portfolio estimators by

- Non-parametric Approach
- Parametric Approach

Locally Stationary Returns

- **Non-Parametric Approach**

$$\hat{\mu}\left(\frac{t}{n}\right) = \frac{1}{b_n n} \sum_{s=[t-b_n n/2]+1}^{[t+b_n n/2]} K\left(\frac{t-s}{b_n n}\right) X(s,n)$$

$$\hat{c}\left(\frac{t}{n},k\right) = \frac{1}{b_n n} \sum_{s=[t-b_n n/2]+1}^{[t+b_n n/2]} K\left(\frac{t-s-k/2}{b_n n}\right)$$

$$\times \left\{ X(s,n) - \hat{\mu}\left(\frac{s}{n}\right) \right\} \left\{ X(s+k,n) - \hat{\mu}\left(\frac{s+k}{n}\right) \right\}'$$

$K(\cdot)$: kernel function , b_n : bandwidth

Locally Stationary Returns

- **Non-Parametric Approach**

Write $\theta = (\ \mu(1)'\ ,\ vech\{c(1,0)\})',\ \hat{\theta} = (\ \hat{\mu}(1)'\ ,\ vech\{\hat{c}(1,0)\}\ )'.$

- Theorem 4

$$\sqrt{b_n n}\left(\hat{\theta} - \theta\right) \xrightarrow{\Lambda} N\left(\Psi_{NP}, \Omega_{NP}\right)$$

and

$$\sqrt{b_n n}\left(g\left(\hat{\theta}\right) - g(\theta)\right) \xrightarrow{\Lambda} N\left(\left(\frac{\partial g}{\partial \theta'}\right)\Psi_{NP}, \left(\frac{\partial g}{\partial \theta'}\right)\Omega_{NP}\left(\frac{\partial g}{\partial \theta'}\right)'\right)$$

Locally Stationary Returns

- **Non-Parametric Approach (simulation result)**

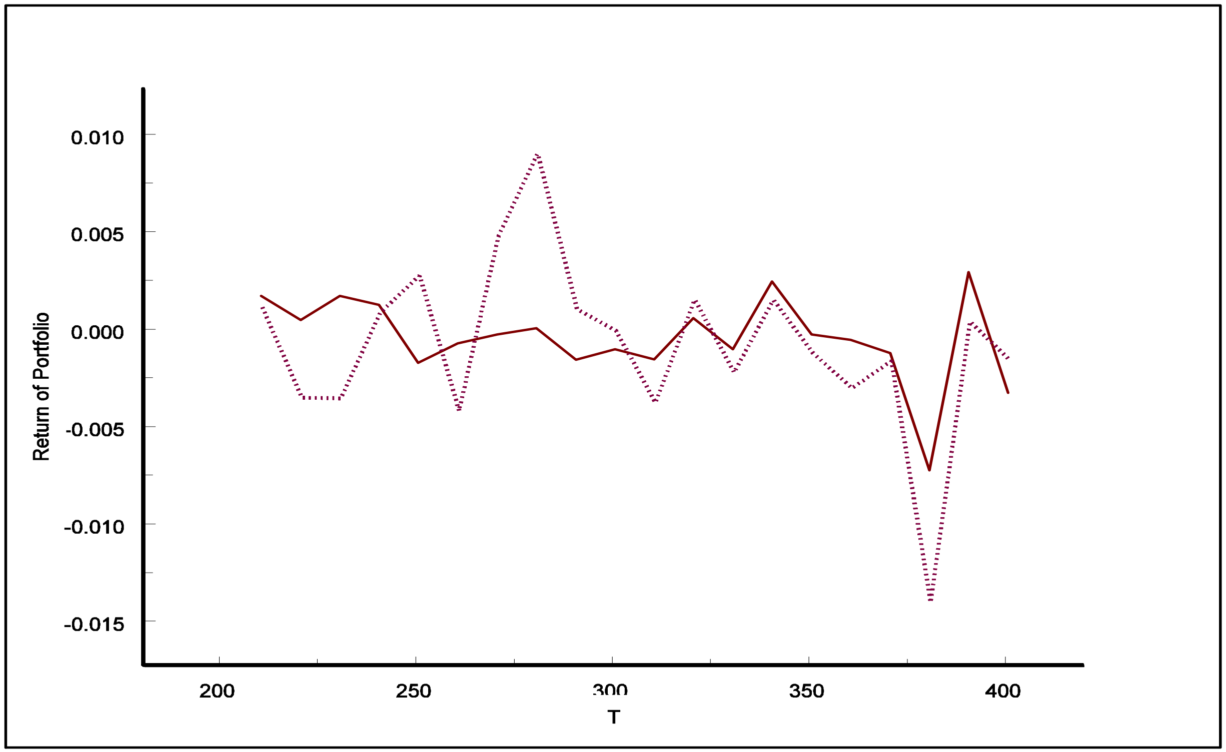
We consider two optimal portfolio estimators by using NIKKEI, HITACHI, HONDA, JT, NOMURA HD, NTT DOCOMO, SEVEN ELEVEN, SOFT BANK and TODEN 's log return from 2000/01/04 to 2001/08/16

$$\hat{g}_n^{kernel} = \frac{\{\hat{c}_n^{\ker}\}^{-1}\mathbf{e}}{\mathbf{e}\{\hat{c}_n^{\ker}\}^{-1}\mathbf{e}}, \quad \hat{g}_{n_0}^{traditional} = \frac{\{\hat{c}_{n_0}^{tra}\}^{-1}\mathbf{e}}{\mathbf{e}\{\hat{c}_{n_0}^{tra}\}^{-1}\mathbf{e}}$$

where $\mathbf{e} = (1,\ldots,1)' \; (10 \times 1 - vector)$

Locally Stationary Returns

- Non-Parametric Approach (simulation result)



- Dotted line : The portfolio return of the traditional estimator
- Solid line : The portfolio return of the kernel estimator

Locally Stationary Returns

- **Parametric Approach**

$H(p;\theta)$: underlying parameter $\theta \in \Theta$ and the probability density of $U(t) = p(\cdot)$.

Define

$$\theta_n = \theta + \frac{1}{\sqrt{n}} h \ , \qquad h = (h_1, \ldots, h_q)' \in \boldsymbol{H} \subset \mathbf{R}^q .$$

For two hypothetical values $\theta, \theta_n \in \Theta$, the log-likelihood ratio is

$$\Lambda_n(\theta, \theta_n) \equiv \log \frac{dP_{\theta_n, n}}{dP_{\theta, n}} = 2 \sum_{l=1}^{n} \log \Phi_{l,n}(\theta, \theta_n) .$$

Locally Stationary Returns

- **Parametric Approach**
- Theorem 5 (LAN theorem)

The sequence of experiments

$$\boldsymbol{E}_n = \left\{ \mathbf{R}^{\mathbf{Z}}, \boldsymbol{B}^{\mathbf{Z}}, \left\{ P_{\theta,n} : \theta \in \Theta \subset \mathbf{R}^{\mathbf{q}} \right\} \right\}, \quad n \in \mathbf{N}$$

is locally asymptotically normal.

(i) For all $\theta \in \Theta$, under $H(p,\theta)$, as $n \to \infty$

$$\Lambda_n(\theta, \theta_n) = \Delta_n(h\,;\,\theta\,) - \frac{1}{2}\Gamma_h(\theta) + o_p(1)\,,$$

(ii) Under $H(p,\theta)$,

$$\Delta_n(h\,;\,\theta\,) \xrightarrow{\Lambda} N(0, \Gamma_h(\theta))\,.$$

Locally Stationary Returns

- **Parametric Approach**

If $\sqrt{n}\left(\hat{\theta}_n - \theta\right) - \Gamma(\theta)^{-1}\Delta_n(\theta) = o_p(1)$ under $H(p,\theta)$ (*)

$$\Downarrow$$

$\hat{\theta}_n$ minimized $E\left[l\left\{\sqrt{n}\left(\hat{\theta}_n - \theta\right)\right\}\right]$ (Asymptotically efficient)
where l is an appropriate loss function.

Let $\hat{\theta}_{QML} \equiv \arg\min_{\theta\in\Theta} \Lambda_n(\theta)$

where $\Lambda_n(\theta)$ is a quasi-likelihood.
Then, $\hat{\theta}_{QML}$ satisfies (*), i.e., **asymptotically efficient**.

Locally Stationary Returns

- **Parametric Approach**
- Theorem 6

$$\sqrt{n}\left(\hat{\theta}_{QML}-\theta\right)\xrightarrow{\Lambda}N\left(0\ ,\ V^{-1}\right)$$

and

$$\sqrt{n}\left(g\left(\mu_{\hat{\theta}_{QML}},c_{\hat{\theta}_{QML}}\right)-g\left(\mu_\theta,c_\theta\right)\right)\xrightarrow{\Lambda}N\left(\ 0,\left(\frac{\partial g}{\partial\theta'}\right)V^{-1}\left(\frac{\partial g}{\partial\theta'}\right)'\right)$$

where $\mu_\theta=\mu_\theta(1),\ c_\theta=c_\theta(1,0)$.

The estimator $g(\mu_{\hat{\theta}_{QML}}\ ,\ c_{\hat{\theta}_{QML}})$ is asymptotically efficient based on LAN.

Optimal Portfolios depending on Higher Order Cumulants

- Under **Non-Gaussianity**, if we use general utility function, we get the optimal portfolio depending on higher order moments of the return.

- Writing
  - Traditional optimal portfolio

$$g_* = g_*(\mu , \Sigma)$$

  - Proposed optimal portfolio

$$g = g(\mu , \Sigma , \textit{third order cumulants})$$

we compare two estimators when the return process is **contiguous** to a Gaussian one.

Optimal Portfolios depending on Higher Order Cumulants

- The expected utility can be represented as;

$$E[U(Y(t))]$$

$$\approx E[c_1^Y(t)] + \frac{1}{2!} D^2 E[c_1^Y(t)] c_2^Y(t) + \frac{1}{3!} D^3 E[c_1^Y(t)] c_3^Y(t)$$

$$= AE[U(Y(t))] \quad (say)$$

where $Y(t) = (\alpha_0, \alpha_1, \ldots, \alpha_p)' X(t)$

- The approximate optimal portfolio may be written as

$$\begin{cases} \max\limits_{\alpha_0, \alpha} AE[U(Y(t))] \\ subject \;\; to \;\; \alpha_0 + \sum_{i=1}^{p} \alpha_i = 1 \end{cases}$$

=> proposed optimal portfolio is the form of

$$g = g(\mu, \Sigma, c_3^Y)$$

Optimal Portfolios depending on Higher Order Cumulants

- **Estimator**

Given $\{X(1), \ldots, X(n)\}$, we introduce

$$\hat{c}^{a_1} = \frac{1}{n}\sum_{s=1}^{n} X_{a_1}(s)$$

$$\hat{c}^{a_2 a_3} = \frac{1}{n}\sum_{s=1}^{n} \{X_{a_2}(s) - \hat{c}^{a_2}\}\{X_{a_3}(s) - \hat{c}^{a_3}\}$$

$$\hat{c}^{a_4 a_5 a_6} = \frac{1}{n}\sum_{s=1}^{n} \{X_{a_4}(s) - \hat{c}^{a_4}\}\{X_{a_5}(s) - \hat{c}^{a_5}\}\{X_{a_6}(s) - \hat{c}^{a_6}\}$$

Optimal Portfolios depending on Higher Order Cumulants

- **CLT**

Write $\theta = (c^{a_1}, c^{a_2 a_3}, c^{a_4 a_5 a_6})', \hat{\theta} = (\hat{c}^{a_1}, \hat{c}^{a_2 a_3}, \hat{c}^{a_4 a_5 a_6})'.$

- Theorem 7

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{\Lambda} N(0, \Omega)$$

and

$$\sqrt{n}\left(g\left(\hat{\theta}\right) - g(\theta)\right) \xrightarrow{\Lambda} N\left(0 , \left(\frac{\partial g}{\partial \theta'}\right) \Omega \left(\frac{\partial g}{\partial \theta'}\right)' \right)$$

Optimal Portfolios depending on Higher Order Cumulants

- **Comparison of two estimators under contiguous assumption**

- Theorem 8

  Under Contiguous Assumption,

$$tr\left[MSE\left\{g\left(\hat{\theta}\right)\right\}\right] \le tr\left[MSE\left\{g_*\left(\hat{\theta}_*\right)\right\}\right]$$

$$\text{if } \; tr\left\{Dg^{a_4a_5a_6}\left(\Omega_{33} - h^{a_4a_5a_6}{}'h^{a_4a_5a_6}\right)Dg^{a_4a_5a_6}\right\} \le 0$$

$$tr\left[MSE\left\{g\left(\hat{\theta}\right)\right\}\right] > tr\left[MSE\left\{g_*\left(\hat{\theta}_*\right)\right\}\right]$$

$$\text{if } \; tr\left\{Dg^{a_4a_5a_6}\left(\Omega_{33} - h^{a_4a_5a_6}{}'h^{a_4a_5a_6}\right)Dg^{a_4a_5a_6}\right\} > 0$$

where Ω_{33} is the sub-matrix of asymptotic covariance matrix Ω .

Conclusion

- Portfolio Estimation：

In the literature => “ i.i.d. returns” or

“i.i.d. primitive estimation”

- Obtained Results：

- Returns => Gaussian Stationary Processes

=> Non Gaussian Stationary Processes

=> Non Gaussian, Locally Stationary Process

- Unified Optimal Estimation Theory for Portfolio was established by use of LAN.

- Portfolio

$$g(\mu , \Sigma) \to g(\mu , \Sigma , \textit{third order cumulants})$$

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